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# Lattice Approach to Wigner-Type Theorems

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The Wigner's Theorem states that a bijective transformation of the set of all one-dimensional linear subspaces of a complex Hilbert space which preserves orthogonality is induced by either a unitary or an anti-unitary operator. There exist many Wigner-type theorems, in particular in indefinite metric spaces, von Neumanns algebras and Banach spaces and we try to find a common origin of all these results by using properties of the lattice subspaces of certain topological vector spaces. We prove a Wigner-type theorem for a pair of dual spaces which allows us to obtain, as particular cases, the usual Wigner's Theorem and some of its generalizations.

**KEY WORDS:** orthomodular lattices; lattices of subspaces; pair of dual spaces; Wigner's theorem.

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## 1. INTRODUCTION

If $H$ is a complex Hilbert space and $S$ a bijective transformation of the set of all one-dimensional linear subspaces of $H$ which preserves angles between any pair of such subspaces then the Wigner's Theorem states that $S$ is induced by either a unitary or an anti-unitary operator on $H$. If $\dim H \geq 3$, U. Uhlhorn improved this result in Uhlhorn (1963) by requiring that $S$ only preserves the orthogonality between the one-dimensional subspaces also called lines in the sequel.

There exist in the literature many generalizations of the Wigner's Theorem, in particular to indefinite metric spaces (Bracci *et al.*, 1975; Molnár, 2002), von Neumann algebras (Molnár, 2000), complex Banach spaces (Molnár, 2002), projections of rank one in Banach spaces (Molnár, 2002) and it seems interesting to find a common origin for all these results. In the present paper, we search the roots of the Wigner's Theorem in a result of Chevalier (2004) which describes automorphisms of the lattice of all closed subspaces of certain topological vector spaces by means of bicontinuous bijections. This result generalizes the First Fundamental

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Theorem of projective geometry related to the automorphisms of the lattice of all subspaces of a vector space.

If we replace the Hilbert space $H$ by a topological vector space $E$ over a field $K$, a first problem is to define an orthogonality relation on the set of all lines of $E$. In general, orthogonality relations on a lattice of subspaces are defined by means of non-degenerate bilinear forms and generally no natural bilinear form is available on $E \times E$. On the other hand, let $E^*$ be the algebraic dual space of $E$ formed by all the linear functionals on $E$. There always exists a natural non-degenerate bilinear form on $E \times E^*$, namely the mapping $B : E \times E^* \to K$ defined by

$$B(x, y) = y(x), \quad x \in E, \quad y \in E^*.$$

Since $E$ is a topological space, closed subspaces seem more convenient than general subspaces and this condition forces to replace $E^*$ by $E'$, the topological dual of $E$ formed by all continuous linear functionals on $E$. But now, the restriction of the bilinear form $B$ to $E \times E'$ is not necessarily non-degenerate and that leads to consider only pairs $(E, E')$ which are pairs of dual spaces in the sense of Mackey (1945) or Dieudonné (1942).

If $(E, F)$ is a pair of dual spaces then the lattice of all closed spaces of $E$ is an irreducible complete DAC-lattice and such lattices appear as the natural setting of the lattice part of this work. In the first section, we will specify the definitions and the main properties of pairs of dual spaces and DAC-lattices.

Section 2 is devoted to the lattice tools necessary to a generalization of the Wigner's Theorem and in Section 3 Wigner-type theorem is proved in the setting of a pair of dual spaces $(E, F)$ but, in this new version of the theorem, one line is a line of $E$ and the other is a line of $F$.

As a consequence of the main result of Section 3, Wigner-type theorems for real locally convex spaces and for complex normed spaces are established in Section 4. If $H$ is a Hilbert space, the identification of $H$ and its topological dual allows one to find again the original Wigner's Theorem.

Information about the lattice concepts used in this paper may be found in Maeda (1970) and Köthe (1969) is a good reference for topological vector spaces.

In the whole of the paper, the dimensions of all the vector spaces are not less than 3 and the heights of all the lattices are not less than 4.

## 2. DAC-LATTICES AND PAIRS OF DUAL SPACES

### 2.1. Definitions and Main Properties

An AC-lattice is an atomistic lattice with the covering property: if $p$ is an atom and $a \wedge p = 0$ then $a \lessdot a \vee p$, that is $a \leq x \leq a \vee p$ implies $a = x$ or $a \vee p = x$. In general, $At(L)$ will denote the set of all atoms of a lattice $L$ and if $L^*$ is the dual lattice of $L$ then $At(L^*)$ is also the set of all coatoms of $L$.

If $L$ and its dual lattice $L^*$ are AC-lattices, $L$ is called a DAC-lattice (Maeda, 1970). Irreducible complete DAC-lattices of heights $\geq 4$ are representable by lattices of closed subspaces and many lattices of subspaces are DAC-lattices. We will now specify this last assertion.

Let $K$ be a field, $E$ a left vector space over $K$, $F$ a right vector space over $K$. If there exists a non-degenerate bilinear form $\mathcal{B}$ on $E \times F$, we say that $(E, F)$ is a pair of dual spaces. Since the form is non-degenerate, $F$ can be interpreted as a subspace of the algebraic dual $E^*$ of $E$ and $E$ as a subspace of $F^*$. This interpretation allows one to write, for any $x \in E$ and any $y \in F$, $x(y) = y(x) = \mathcal{B}(x, y)$.

For example, if $E$ is a locally convex space and $E'$ its topological dual space then $(E, E')$ is naturally a pair of dual spaces with $\mathcal{B}(x, y) = y(x)$ (Köthe, 1969, p. 234).

For a subspace $A$ of $E$, we put

$$A^{\perp} = \{y \in F \mid \mathcal{B}(x, y) = 0 \text{ for every } x \in A\}.$$

Similarly, let

$$B^{\perp} = \{x \in E \mid \mathcal{B}(x, y) = 0 \text{ for every } y \in B\}$$

for every subspace $B$ of $F$. A subspace $A$ of $E$ is called $F$-closed if $A = A^{\perp\perp}$ and the set of all $F$-closed subspaces, denoted by $L_F(E)$ and ordered by set-inclusion, is a complete irreducible DAC-lattice (Maeda, 1970, Theorem 33.4). Conversely, for any irreducible complete DAC-lattice $L$ of height $\geq 4$, there exists a pair $(E, F)$ of dual spaces such that $L$ is isomorphic to the lattice of all $F$-closed subspaces of $E$ (Maeda, 1970, Theorem 33.7, Köthe 1969, Section 10.3).

The set $L_E(F)$ of all $E$-closed subspaces of $F$ is similarly defined and is also a DAC-lattice. The two DAC-lattices $L_F(E)$ and $L_E(F)$ are dual isomorphic by the mapping $A \to A^{\perp}$ (Maeda, 1970, Theorem 33.4) and an element $X$ of $L_F(E)$ and an element $Y$ of $L_E(F)$ are said to be orthogonal if $X \subset Y^{\perp}$ (Equivalently, $Y \subset X^{\perp}$) and we write $X \perp Y$.

Let $(E, F)$ be a pair of dual spaces. The linear weak topology on $E$, denoted by $\sigma(E, F)$, is the linear topology defined by taking $\{G^{\perp} \mid G \subset F,\ \dim G < \infty\}$ as a basis of neighbourhoods of 0. If $F$ is interpreted as a subspace of the algebraic dual of $E$ then a sub-basis of neighbourhoods of 0 consists of kernels of elements of $F$.

The linear weak topology on $F$, noted $\sigma(F, E)$, is defined in the same way. The space $F$ can be interpreted as the topological dual of $E$ for the $\sigma(E, F)$ topology and $E$ as the topological dual of $F$ for the $\sigma(F, E)$ topology. Equipped with their linear weak topologies, $E$ and $F$ are topological vector spaces (Köthe, 1969, Section 10.3) if the topology on $K$ is discrete.

Moreover, for a subspace $G \subset E$, we have $\overline{G} = G^{\perp\perp}$ and thus to be a closed subspace in $E$ is an unambiguous notion. If $K = \mathbb{R}$ or $\mathbb{C}$, this result generalizes to any pair $(E, F)$ and any locally convex topology over $E$ when $F$ is the dual of $E$ for this topology (Köthe, 1969, Section 20.3).

## 2.2. The Adjoint of a Semi-Linear Map

Let $(E, F)$ be a pair of dual spaces and $f : E \to E$ a $\tau$-linear mapping, that is a group homomorphism satisfying $f(\lambda x) = \tau(\lambda) f(x)$ where $\lambda \in K$, $x \in E$ and $\tau$ is an automorphism of $K$.

If $y$ is an element of $E^*$ then the mapping $x \in E \to \tau^{-1}(y(f(x)))$ belongs to $E^*$ since

$$\tau^{-1}(y(f(x + x'))) = \tau^{-1}(y(f(x))) + \tau^{-1}(y(f(x')))$$

and, for any $\lambda \in K$,

$$\tau^{-1}(y(f(\lambda x))) = \tau^{-1}(y(\tau(\lambda) f(x))) = \tau^{-1}(\tau(\lambda) y(f(x))) = \lambda \tau^{-1}(y(f(x))).$$

Let us define $f^* : E^* \to E^*$ by $f^*(y)(x) = \tau^{-1}(y(f(x)))$ for any $y \in E^*$ and any $x \in E$. For $y,\ y' \in E^*$ and $\lambda \in K$, we have $f^*(y + y') = f^*(y) + f^*(y')$, and

$$\begin{aligned} f^*(y\lambda)(x) &= \tau^{-1}[(y\lambda)(f(x))] = \tau^{-1}[y(f(x))\lambda] \\ &= \tau^{-1}(y(f(x)))\tau^{-1}(\lambda) = f^*(y)(x)\tau^{-1}(\lambda) \end{aligned}$$

and so $f^*(y\lambda) = f^*(y)\tau^{-1}(\lambda)$. The mapping $f^*$ is $\tau^{-1}$-linear and will be called the adjoint of $f$.

Assume that $f$ is weakly continuous. If $y \in F \subset E^*$ then the mapping $x \in E \to \tau^{-1}(y(f(x)))$ is weakly continuous and therefore $f^*(y) \in F$. We will prove that the restriction of $f^*$ to $F$ is weakly continuous.

Let $x \in E = F'$, $x \neq 0$. As $x$ and $f(x)$ can be identified to continuous linear functionals on $F$, we can write, for any $y \in F$,

$$\begin{aligned} y \in \ker f(x) &\Leftrightarrow (f(x))(y) = 0 \Leftrightarrow y(f(x)) = 0 \Leftrightarrow \tau^{-1}(y(f(x))) = 0 \\ &\Leftrightarrow (f^*(y))(x) = 0 \Leftrightarrow f^*(y) \in \ker x \end{aligned}$$

and therefore, $f^*(\ker f(x)) \subset \ker x$. Thus, for any element $U = \ker x$ of a sub-basis of neighbourhoods of $0 \in F = E'$ there exists a neighbourhhood $V = \ker f(x)$ of $0 \in F$ such that $f^*(V) \subset U$. The mapping $f^*$ is weakly continuous at 0 and consequently, since $f^*$ is a homomorphism of topological groups, it is weakly continuous on $F$.

In what follows, if $f : E \to E$ is a weakly continuous $\tau$-linear mapping then $f^*$ will always mean the restriction of $f^*$ to $F \subset E^*$ and thus $f^{**}$ is a mapping from $E$ to $E$.

Now let us consider a closed subspace $X$ of $E$. For any $x \in X$ and any $y \in F$, $y(f(x)) = 0$ is equivalent to $f^*(y)(x) = 0$ and so, as for linear mappings,

$f^{*-1}(X^{\perp}) = f(X)^{\perp}$ for any $X \in L_F(E)$. Others results about adjoints are: $f^{**} = f$ for a weakly continuous semi-linear mapping and $f^{*-1} = f^{-1*}$ if $f$ is a weakly continuous semi-linear bijection with a weakly continuous inverse. A proof of the second claim is as follows.

If for $y_1, y_2 \in F$, $f^*(y_1) = f^*(y_2)$ then for any $x \in E$, $f^*(y_1)(x) = f^*(y_2)(x)$ and, since $\tau$ is an automorphism, $y_1(f(x)) = y_2(f(x))$. Since $f$ is onto, $y_1 = y_2$ and $f^*$ is one-to-one.

Now let $z \in F$. Define $y \in F$ by $y(x) = \tau(z(f^{-1}(x)))$. For $x \in E$, we have :

$$f^*(y)(x) = \tau^{-1}(y(f(x))) = \tau^{-1}(\tau(z(f^{-1}(f(x))))) = z(x).$$

Thus, $f^*(y) = z$ and $f^*$ is onto. The mapping $f^*$ is bijective and we will prove that $f^{-1*} = f^{*-1}$.

For $x \in E$ and $y \in F$, we have :

$$f^*(f^{-1*}(y))(x) = \tau^{-1}((f^{-1*}(y))(f(x))) = \tau^{-1}(\tau(y(f^{-1}(f(x))))) = y(x).$$

Therefore, $f^*(f^{-1*}(y)) = y$ holds. Similarly,

$$f^{-1*}((f^*(y))(x) = \tau(f^*(y))(f^{-1}(x))) = \tau(\tau^{-1}(y(f(f^{-1}(x))))) = y(x).$$

Hence, $f^{-1*}(f^*(y)) = y$ and the proof is complete.

## 3. THE LATTICE TOOLS FOR A LATTICE APPROACH TO THE WIGNER'S THEOREM

**Proposition 1.** *Let $L$ be a complete DAC-lattice. If $f$ is an automorphism of the poset $At(L) \cup At(L^*)$ then $f$ extends to an automorphism $\phi$ of the lattice $L$.*

**Proof:** Clearly, $f$ is a bijection of $At(L)$ and a bijection of $At(L^*)$.

Let $\bigvee_{i \in I} p_i$ and $\bigvee_{j \in J} q_j$ be two joins of atoms of $L$. We will prove that

$$\bigvee_{i \in I} p_i \leq \bigvee_{j \in J} q_j \Leftrightarrow \bigvee_{i \in I} f(p_i) \leq \bigvee_{j \in J} f(q_j).$$

Assume $\bigvee_{i \in I} p_i \leq \bigvee_{j \in J} q_j$ and let $P = f(Q)$ a coatom of $L$. We have :

$$\begin{aligned}\bigvee_{j \in J} f(q_j) \leq P = f(Q) &\Leftrightarrow \forall j \in J,\ f(q_j) \leq f(Q)\\ &\Leftrightarrow \forall j \in J,\ q_j \leq Q\\ &\Leftrightarrow \bigvee_{j \in J} q_j \leq Q\end{aligned}$$

$$\begin{aligned}
&\Rightarrow \bigvee_{i\in I} p_i \leq Q \\
&\Leftrightarrow \forall i \in I,\ p_i \leq Q \\
&\Leftrightarrow \forall i \in I,\ f(p_i) \leq f(Q) = P \\
&\Leftrightarrow \bigvee_{i\in I} f(p_i) \leq P,
\end{aligned}$$

and thus $\bigvee_{i\in I} f(p_i) \leq \bigvee_{j\in J} f(q_j)$.

Now, assume $\bigvee_{i\in I} f(p_i) \leq \bigvee_{j\in J} f(q_j)$. Since $f^{-1}$ is also an automorphism of $At(L) \cup At(L^*)$ we have, by the previous part of the proof, $\bigvee_{i\in I} p_i \leq \bigvee_{j\in J} q_j$.

In particular,

$$\bigvee_{i\in I} p_i = \bigvee_{j\in J} q_j \Leftrightarrow \bigvee_{i\in I} f(p_i) = \bigvee_{j\in J} f(q_j)$$

and so we can define an extension $\phi$ of $f$ to $L$ by

$$\phi(x) = \bigvee_{i\in I} f(p_i)$$

if $0 \neq x = \bigvee_{i\in I} p_i$ and $\phi(0) = 0$. The mapping $\phi$ is an automorphism of the lattice $L$ which extends $f$. □

The following result generalizes the First Fundamental Theorem of projective geometry (Baer, 1952) to lattices of closed subspaces.

**Proposition 2.** *Let $(E_1, F_1)$ and $(E_2, F_2)$ be two pairs of dual spaces over the fields $K_1$ and $K_2$. If there exists an isomorphism $\psi$ of the lattice $L_{F_1}(E_1)$ onto the lattice $L_{F_2}(E_2)$ then $K_1$ and $K_2$ are isomorphic fields and there exists a bicontinuous semi-linear bijection $s : E_1 \mapsto E_2$ such that, for every $F_1$-closed subspace $M$ of $E_1$, $\psi(M) = s(M)$. If a bicontinuous $\tau$-linear bijection $s$ and a bicontinuous $\tau'$-linear bijection $s'$ generate the same automorphism $\psi$ then there exists $k \in K_2$ such that $\tau' = k\tau k^{-1}$ and $s' = ks$.*

**Proof:** The first part of the proposition is proved in Chevalier (2004, Proposition 3) and the second part needs a proof only if $E_1$ is infinite dimensional.

Assume that $s$ and $s'$ generate $\psi$ and let $M$ be a subspace of $E_1$ such that $3 \leq \dim M < \infty$. We have $\psi(M) = s(M) = s'(M)$ and by the First Fundamental Theorem of projective geometry, there exists $k \in K_2$ such that, for any $x \in M$, $s'(x) = ks(x)$ and $\tau' = k\tau k^{-1}$. Let $y \in E_1/M$ and $N = M \oplus K_1 y$. There exists $k' \in K_2$ such that, for any $x \in N$, $s'(x) = k's(x)$. If $0 \neq x \in M$ then $0 \neq s(x) = ks'(x) = k's'(x)$ and thus $k = k'$. Therefore, $s'(y) = ks(y)$ and $s' = ks$. □

## 4. WIGNER-TYPE THEOREMS

**Proposition 3.** *(A Wigner-type theorem for DAC-lattices) Let $L$ be an irreducible complete DAC-lattice and $f$ an automorphism of the poset $At(L) \cup At(L^*)$. If $L$ is representable as the lattice $L_F(E)$ of all $F$-closed subspaces of a pair of dual spaces $(E, F)$ then $f$ extends to an automorphism $\phi$ of $L_F(E)$ and there exists a bicontinuous semi-linear bijection $s : E \to E$ such that $\phi(M) = s(M)$ for all $M \in L_F(E)$.*

**Proof:** Use Propositions (1) and (2).

**Remark.** Let $L$ be the lattice of all subspaces of a vector space $E$. If $f$ is an automorphism of $At(L) \cup At(L^*)$ (roughly speaking, $f$ preserves in both directions inclusion of lines in hyperplanes) then $f$ extends to an automorphism $\Phi$ of $L$ and there exists a semi-linear bijection $s : E \to E$ such that, for any subspace $X$, $f(X) = s(X)$.

Let $(E, F)$ be a pair of dual spaces. If $f : At(L_F(E)) \cup At(L_E(F)) \to At(L_F(E)) \cup At(L_E(F))$ is at the same time a bijection of $At(L_F(E))$ and a bijection of $At(L_E(F))$ such that, for any $p \in At(L_F(E))$ and any $q \in At(L_E(F))$,

$$p \perp q \Leftrightarrow f(p) \perp f(q)$$

then $f$ is called a Wigner bijection over $(E, F)$. □

**Proposition 4.** *(A Wigner-type theorem for a pair of dual spaces) Let $f$ be a Wigner bijection over a pair $(E, F)$ of dual spaces. There exists a bicontinuous semi-linear bijection $s : E \to E$ such that:*

1. *for any $p \in At(L_F(E))$, $f(p) = s(p)$,*
2. *for any $q \in At(L_E(F))$, $f(q) = s^{*-1}(q)$.*

**Proof:** As $M \in L_F(E) \to M^\perp \in L_E(F)$ is an anti-isomorphism of lattices, we can define a bijection $f_1$ of $At(L_F(E)^*)$ by $f_1(P) = f(P^\perp)^\perp$ for any $P \in At(L_F(E)^*)$. Let $g$ be the extension of $f_1$ to $At(L_F(E)) \cup At(L_F(E)^*)$ which agrees with $f$ on $At(L_F(E))$. If $p \in At(L_F(E))$ and $P \in At(L_F(E)^*)$, we have $p \leq P$ if and only if $p \perp P^\perp$ which is also equivalent to $f(p) \perp f(P^\perp) = g(P)^\perp$ that is $g(p) \leq g(P)$.

By Proposition (1), $g$ extends to an automorphism $G$ of the lattice $L_F(E)$ and by using Proposition (3) there exists a bicontinuous semi-linear bijection $s$ such that, for every $F$-closed subspace $M$, $G(M) = s(M)$. In particular, for every atom $p$ of $L_F(E)$, $s(p) = G(p) = g(p) = f(p)$.

Let $q \in At(L_E(F))$. We have :

$$s^{*-1}(q) = s(q^\perp)^\perp = f_1(q^\perp)^\perp = f(q). \qquad \square$$

**Remark.** The correspondences $G : X \in L_F(E) \to s(X) \in L_F(E)$ and $H : Y \in L_E(F) \to s^{*-1}(X) \in L_E(F)$ are automorphisms of the DAC-lattices $L_F(E)$ and $L_E(F)$. The pair $(G, H)$ preserves orthogonality of closed subspaces :

$$X \in L_F(E) \perp Y \in L_E(F) \Leftrightarrow G(X) \perp H(Y).$$

Moreover, for any $X \in L_F(E)$, $G(X) = H(X^\perp)^\perp$.

## 5. EXAMPLES

A linear mapping $f$, defined on a locally convex space $E$ over $K = \mathbb{R}$ or $\mathbb{C}$ is weakly continuous if and only if $f$ is continuous with respect to the linear weak topology $\sigma(E, E')$ (Köthe, 1969, 20.4). If $K = \mathbb{R}$ then a semi-linear mapping is linear since the identity is the only automorphism of $\mathbb{R}$ and we have the following version of the Wigner's Theorem.

**Corollary 1.** *Let $E$ be a real locally convex space and $E'$ its dual. If $f$ is a Wigner bijection over the dual pair $(E, E')$ then there exists a weakly bicontinuous linear bijection $s : E \to E$ such that*

- *for any $p \in At(L_{E'}(E))$, $f(p) = s(p)$,*
- *for any $q \in At(L_E(E'))$, $f(q) = s^{*-1}(q)$.*

*If $E$ is metrizable then $s$ is continuous.*

For the last claim of this corollary, we have used the fact that weakly continuous linear mappings between metrizable spaces are continuous (Schaefer, 1966, Chapter IV, 3.4 and 7.4).

If $K = \mathbb{C}$ then the automorphism $\tau$ associated to the semi-linear bijection $s$ of Proposition (4) can be not continuous (in a locally convex space over a field $K$, the topology on $K$ is not the discrete one but is defined by means of the modulus) and an extra hypothesis seems necessary to obtain a Wigner's Theorem close to the classical one.

**Corollary 2.** *Let $E$ be an infinite-dimensional complex normed space and $f$ a Wigner bijection over the dual pair $(E, E')$. There exists a linear bijection or a conjugate linear bijection $s : E \to E$ which is bicontinuous for the norm topology and such that:*

- *for any $p \in At(L_{E'}(E))$, $f(p) = s(p)$,*
- *for any $q \in At(L_E(E')$, $f(q) = s^{*-1}(q)$.*

**Proof:** Let $s$ be the semi-linear bijection obtained by using Proposition (4). Since $s$ is continuous for the weak linear topology, $s$ carries orthogonally closed hyperplanes to orthogonally closed hyperplanes (Chevalier, 2004, Proposition 3). But orthogonally closed subspaces of $E$ agree with topologically closed subspaces (Köthe, 1969, Section 20, 3 (2)) and by using a result of Kakutani (1946) or Fillmore *et al.* (1984), Lemma 2, $s$ is either linear or conjugate linear. A linear mapping on a metrizable space $E$ is continuous if and only if this mapping is continuous for the linear weak topology $\sigma(E, E')$ and the generalization of this result to a conjugate linear mapping is easy. Thus, $s$ is continuous and by, using a similar proof, $s^{-1}$ is also continuous. □

**Remark.** In Molnár 2002, L. Molnár proved the same result for complex Banach spaces.

The following corollary is the classical Wigner's Theorem and here its interest is only its proof which uses the previous results and specially the Wigner-type theorem for pairs of dual spaces.

**Corollary 3.** *Let $H$ be a Hilbert space over $K = \mathbb{R}$ or $\mathbb{C}$ and $f$ a bijection of the set of all lines of $H$ such that*

$$p \perp q \Leftrightarrow f(p) \perp f(q).$$

*The mapping $f$ extends to an automorphism $\phi$ of the orthomodular lattice of all closed subspaces of $H$ and there exists a weakly bicontinuous semi-linear mapping $r : H \to H$ such that $r^* = r^{-1}$ and, for any closed subspace $M$ of $H$, $r(M) = \phi(M)$. Moreover :*

1. *if $K = \mathbb{R}$ then $r$ is a unitary operator,*
2. *If $K = \mathbb{C}$ and $H$ infinite dimensional, then $r$ is either a unitary or an anti-unitary operator.*

**Proof:** Since $H$ is a Hilbert space, the correspondence which associates to $y \in H$ the continuous functional $\lambda_y : x \to \langle x, y\rangle$ is an anti-isomorphism of $H$ onto its dual. This anti-isomorphism allows one to identify the lattice of all closed subspaces of $H$ and the lattice of all closed subspaces of its dual and to extends $f$ to a Wigner bijection on the dual pair $(H, H)$. Let $s : H \to H$ be the semi-linear bijection obtained by using Proposition (4). For any closed subspace $X$ of $H$, we have $s(X) = s^{*-1}(X)$. Citing Proposition (2), there exists $\lambda \in K$ such that $s = \lambda s^{*-1}$. We have $ss^* = \lambda 1_{\mathbb{C}}$ and therefore $\lambda > 0$. If $r = (1)/(\sqrt{\lambda})s$ then $r$ and $s$ generate the same automorphism $\phi$ of the lattice of all closed subspaces of $H$ and $r^{-1} = r^*$.

By using the general relation $r^{*-1}(X^{\perp}) = r(X)^{\perp}$ and $r^{*-1} = r$ , $\phi(X^{\perp}) = \phi(X)^{\perp}$ holds and $\phi$ is an automorphism of the orthomodular lattice of all closed subspaces of $H$.

1) If $K = \mathbb{R}$ then, by Corollary 1, $r$ is linear and bicontinuous for the norm topology. Since $r^{-1} = r^*$, $r$ is a unitary operator.
2) If $= \mathbb{C}$, then by Corollary 2, $r$ is linear or conjugate linear, bicontinuous for the norm topology and $r^{-1} = r^*$ implies that $r$ is a unitary operator. □

**Remark.** If $H$ is a finite dimensional complex Hilbert space of dimension not less than 3 then the mapping $r$ is also a unitary or an anti-unitary operator but we don't know a short proof using the results of this paper. For a proof in the more general setting of an indefinite inner product space, see Molnár (2002), Corollary 2.

## 6. CONCLUDING QUESTIONS

(1) In Molnár (2002), the author proved a Wigner-type theorem in the context of an indefinite inner product space. Does Proposition (4) allows one to obtain a short proof of this result?
(2) There exist Wigner-type theorems for projections (Molnár, 2002; Chevalier, 2004b). Since projections (i.e. bounded linear idempotent operators) defined on a Hilbert space does not form a DAC-lattice but an atomistic orthomodular poset, does there exist a result similar to Proposition (2) for this kind of poset?

## REFERENCES

Baer, R. (1952). *Linear Algebra and Projective Geometry*, Academic Press, New York.
Bracci, L., Morchio, G., and Strocchi, F. (1975). Wigner's theorem on symmetries in indefinite metric spaces, *Communications of Mathematical Physics* **41**, 289–299.
Chevalier, G. (2004a). Automorphisms of an orthomodular poset of projections. *International Journal of Theoretical Physics*, **44**, 985–998.
Chevalier, G. (2004b). Wigner-type theorems for projections, manuscript in preparation.

Dieudonné, J. (1942). La dualité dans les espaces vectoriels topologiques. *Annales Scientifiques Ecole Normale Supérieure* **59**, 107–139.

Fillmore, P. A. and Longstaff, W. E. (1984). Isomorphisms of lattices of closed subspaces. *Canadian Journal of Mathematics* **XXXVI**(5), 820–829.

Köthe, G. (1969). *Topological Vector Spaces 1*, Springer-Verlag, Berlin.

Kakutani, S. and Mackey, G. (1946). Ring and lattice characterizations of complex Hilbert spaces. *Bulletin of American Society* **52**, 727–733.

Maeda, F. and Maeda, S. (1970). *Theory of Symmetric Lattices*, Springer-Verlag, Berlin.

Mackey, G. (1945). On infinite-dimensional linear spaces. *Transactions of American Mathematical Society* **57**, 155–207.

Molnár, L. (2000). A Wigner-type theorem on symmetry transformations in type II factors. *International Journal of Theoretical Physics* **39**, 1463–1466.

Molnár, L. (2002). Orthogonality preserving transformations on indefinite inner products: Generalization of Uhlhorn's version of Wigner's Theorem. *Journal of Functional Analysis* **194**, 248–262.

Schaefer, H. H. (1966). *Topological Vector Spaces*, Macmillan, New York.

Uhlhorn, U. (1963). Representation of symmetry transformations in quantum mechanics. *Arkiv för Fysik* **23**, 307–340.